Fermilab FERMILAB-Pub-00/072-T July 2000

Preserving High Multibunch Luminosity in Linear Colliders

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(July 26, 2000)

Abstract

An analytic theory of cumulative multibunch beam breakup in linear colliders is developed. Included is a linear variation of transverse focusing across the bunch train as might be applied, *e.g.*, by chirping the radiofrequency power sources or by using radiofrequency quadrupole magnets. The focusing variation saturates the exponential growth of the beam breakup and establishes an algebraic decay of the transverse bunch displacement versus bunch number. A closed-form expression for the transverse displacement is developed. It is used to quantify the total normalized emittance and thereby isolate the region of parameter space corresponding to high multibunch luminosity.

29.27.Bd, 41.75.Fr

To be useful for high-energy physics, an e^+e^- linear collider must deliver high-energy, high-luminosity beams to the interaction point. The luminosity scales as $P_b/(\mathcal{E}_b\tilde{\varepsilon}_y^{1/2})$ [1], in which P_b, $\mathcal{E}_b$, and $\tilde{\varepsilon}_y$ denote the e^+e^- beams' power, energy, and root-mean-square (rms) normalized vertical emittance, respectively. Consequently, attaining the required high luminosity involves rigorously controlling the quality of high-current beams. This is especially true concerning $\tilde{\varepsilon}_y$. For example, the Next Linear Collider (NLC) design comprises two main linear accelerators (linacs), one for e^+ and one for e^-, each delivering a flat 1 TeV beam, with horizontal and vertical emittances $\tilde{\varepsilon}_x \sim 4$ μm and $\tilde{\varepsilon}_y \sim 0.1$ μm, respectively [2]. The linacs are long ($\sim$10 km) to achieve the high final energy, and concern over beam instabilities is correspondingly heightened.

One worrisome instability is that due to cumulative multibunch beam breakup (BBU). It arises from beam-excited transverse wakes in the accelerating radiofrequency (rf) cavities. Specifically, an imperfectly injected initial bunch excites one or more deflecting modes in the first rf cavity, which then deflect trailing bunches by amounts that depend on their phases relative to the deflecting modes. Trailing bunches that are deflected further away from the beam axis can couple more strongly to these modes in downstream cavities so that the influence of the deflecting modes on the bunch train grows as it moves down the linac. The instability is thereby "cumulative", and it is "multibunch" in the sense that leading bunches influence trailing bunches by way of the deflecting wake. The bunch train exiting the linac is transversely enlarged; its projected emittance (and therefore luminosity) is degraded.

Continuing with the NLC as an example, the main linacs each consist of several thousand accelerating X-band (11.424 GHz) cells. The cells are assembled into arrays of 206-cell Rounded Damped Detuned Structures (RDDS) [3] specially designed to keep long-range transverse wakes small, so that the RDDS is effectively the "fundamental" accelerating unit of the linac. A prototypical RDDS wake amplitude is illustrated in Fig. 1. It arises from a distribution of deflecting modes. To a reasonable approximation the wake may be modeled as a single deflecting dipole mode of representative angular frequency ω, nominally the center frequency in the distribution (about 15 GHz for Fig. 1), and representative quality factor Q.

The NLC bunch train consists of 90 bunches spaced τ=2.8 ns apart and is therefore about 250 ns long. Thus, the effective Q is infinite over most of the bunch train, though the wake does fall off during the tail. In keeping with Fig. 1, the following development is based on a single deflecting dipole mode representative of that found not in a single rf cell, but rather in a complete RDDS. Specifically, the wake is taken to be $w(\zeta) = w_0 \Theta(\zeta) e^{-\zeta/2Q} \sin\zeta$, in which w_0 is the wake amplitude, $\Theta(\zeta)$ is the unit step function, and $\zeta/\omega = t - s/c$ is the time measured after the arrival of the first bunch at position s along the linac. One cannot purposely zero the deflecting-wake kick by putting the bunches at wake zero-crossings because RDDS disallows actively adjusting a deflecting mode to make its frequency a multiple of the accelerating-mode frequency.

In a "continuum approximation" in which the discrete transverse kicks imparted by the rf structures are smoothed along the linac, the equation of transverse motion is [4]:

$$\left[\frac{1}{\gamma}\frac{\partial}{\partial\sigma}\left(\gamma\frac{\partial}{\partial\sigma}\right) + \kappa^2\right] x(\sigma,\zeta) = \epsilon \int_0^\zeta d\zeta'\, w(\zeta-\zeta')\, F(\zeta')\, x(\sigma,\zeta')\ . \qquad (1)$$

Here, $\sigma = s/\mathcal{L}$ denotes location along the linac normalized to the total linac length $\mathcal{L}$, *i.e.*, $0 \le \sigma \le 1$; κ is the net transverse focusing wavenumber multiplied by $\mathcal{L}$; x is the transverse displacement of the beam centroid from the axis; $F(\zeta) = I(\zeta)/\bar{I}$ is a form factor involving the current $I(\zeta)$ and its time-average $\bar{I}$; and $\epsilon = w_0 q e \mathcal{L}^2/(\gamma m c^2 \omega\tau)$ is the dimensionless BBU coupling strength, in which q denotes the bunch charge, and e and $\gamma m c^2$ denote the electron charge and total energy, respectively. Table 1 lists nominal linear-collider parameters, inputs we use for numerical examples.

One method for actively mitigating BBU is to vary the focusing strength along the bunch train [5]. Ideally this would remove the time dependence so that $x(\sigma,\zeta) \to x(\sigma)$, in which case the required focusing strength is, from Eq. (1), $\kappa^2(\sigma,\zeta) = \kappa^2(\sigma) + \epsilon(\sigma)\int_0^\zeta d\zeta' w(\zeta - \zeta')F(\zeta')$. However, because the wake varies rapidly, one cannot achieve this requirement in practice. Instead, by chirping the rf power input to the cavities, or by using rf-quadrupole magnets, one can affect a simpler variation, for example a linear variation in time: $\kappa(\sigma,\zeta) = \kappa(\sigma) + \kappa_{,\zeta}(\sigma,0)\zeta$. In detailed simulations of a contemporary NLC design, Stupakov found that a small (%-level) linear variation could substantially damp multibunch BBU [6].

Our principal goal here is analytically to quantify and explain the benefit of a linear focusing variation. After summarizing the procedure for solving Eq. (1), we write a closed-form solution for the transverse displacement, use it to calculate the total emittance, and then use the emittance to isolate the region of parameter space corresponding to viable linear-collider designs. In developing the solution we incorporate approximations appropriate to a linear-collider design: zero-length (δ-function) bunches, adiabatic variation of parameters along the linac, and strong focusing. Specifically, we take the dependence of focusing strength on beam energy to be $\kappa \propto \gamma^{-1/2}$, a good model for the NLC lattice and one that lends itself to analytic treatment. The first step is to rewrite Eq. (1) in terms of a "chirp-modified" wake. With a new variable $\xi(\sigma,\zeta) \equiv \sqrt{\gamma(\sigma)}x(\sigma,\zeta)e^{-i\zeta\Delta(\sigma)}$, wherein $\Delta(\sigma) \equiv \int_0^\sigma d\sigma' \kappa_{,\zeta}(\sigma',0)$, and with strong focusing, Eq. (1) takes a spatially harmonic form:

$$\left[\frac{\partial^2}{\partial\sigma^2} + \kappa^2(\sigma)\right]\xi(\sigma,\zeta) \simeq \epsilon(\sigma)\int_0^\zeta d\zeta' w_\Delta(\sigma,\zeta-\zeta')F(\zeta')\xi(\sigma,\zeta') \, , \tag{2}$$

the chirp-modified wake being $w_\Delta(\sigma,\zeta) \equiv w(\zeta)e^{-i\zeta\Delta(\sigma)}$. This is an "eikonal approximation" [7] with a subtlety: if the focusing chirp were established through an energy spread, then $\gamma(\sigma) \to \gamma(\sigma,\zeta)$, and a factor $\gamma^{-1/2}(\sigma,\zeta')$ would be trapped in the integration over ζ'. Taking this factor out of the integral then makes Eq. (2) only a model, but one consistent with strong focusing in the sense that the desired energy spread will be commensurate with ϵ, a quantity that is small compared to κ^2. Noting that w_Δ introduces a complex effective Q, namely $(2Q_{eff})^{-1} = (2Q)^{-1} + i\Delta$, one sees immediately that the chirp is important if Q is high, but is masked (and not needed) if Q is sufficiently low.

A formal solution for $\xi(\sigma,\zeta)$, and in turn for the displacement $x_M(\sigma) \equiv x(\sigma,\zeta = M\omega\tau)$ of the M^{th} bunch, is obtained by Fourier-transforming Eq. (2) in time, solving the transformed equation with the WKBJ method as is appropriate for adiabaticity, and Fourier-inverting the solution [4,8]:

$$x_M(\sigma) = \frac{1}{2\pi}\sum_{m=0}^{M} e^{-m\frac{\omega\tau}{2Q}}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta}\left\{x_{M-m}(0)\mathcal{C}(\sigma,\theta;M) + x'_{M-m}(0)\frac{\mathcal{S}(\sigma,\theta;M)}{\Lambda(0,\theta)}\right\} \, , \tag{3}$$

in which

$$\Lambda(\sigma,\theta) \equiv \kappa(\sigma)\left\{1 - \frac{\epsilon(\sigma)}{4\kappa^2(\sigma)}\,\frac{\omega\tau\sin\omega\tau}{\cos[\theta+\omega\tau\Delta(\sigma)]-\cos\omega\tau}\right\} \tag{4}$$

is an auxiliary function reflecting the coupling between the bunch spacing and the deflecting-mode frequency, and

$$\left\{\begin{array}{c} \mathcal{C}(\sigma,\theta;M) \\ \mathcal{S}(\sigma,\theta;M)] \end{array}\right\} \equiv \sqrt{\frac{\Lambda(0,\theta)}{\Lambda(\sigma,\theta)}}\left\{\begin{array}{c} \Re \\ \Im \end{array}\right\}\exp\left[iM\omega\tau\Delta(\sigma) + \int_0^\sigma d\sigma'\Lambda(\sigma',\theta)\right] \tag{5}$$

are cosine-like and sine-like functionals, respectively. The algebraic sign of $\Delta(\sigma)$ affects only the phase of $x_M(\sigma)$; that this is so can be seen from Eq. (3) upon taking $\theta \to -\theta$ and remembering that x_M is real. For the envelope bounding the tranverse amplitudes, the effect of a linear increase in focusing from head to tail is the same as a linear decrease. Moreover, with $\kappa \propto \gamma^{-1/2}$ and $\epsilon \propto \gamma^{-1}$, it is easy to treat arbitrary acceleration, *viz.*, arbitrary $\gamma(\sigma)$. The injection offsets $x_M(0)$ and angles $x'_M(0)$ are also arbitrary; what follows applies, for concreteness, to a misaligned beam for which $x_M(0) = x_0$ and $x'_M(0) = 0$ for every bunch M.

We decompose the sum in Eq. (3) into two parts: $\sum_0^M = \sum_0^\infty - \sum_M^\infty$. The first part pertains to the "steady-state" displacement x_{ss} that would arise were the deflecting wake first seeded with an infinitely long bunch train immediately preceding the actual bunch train. Given strong focusing, the steady-state displacement is

$$x_{ss}(\sigma, M\omega\tau) \simeq x_0\left[\frac{\gamma(0)}{\gamma(\sigma)}\right]^{1/4}\cos\left[M\omega\tau\Delta(\sigma) + \int_0^\sigma d\sigma'\kappa(\sigma')\right]\ ; \tag{6}$$

a nonzero focusing variation establishes a harmonic dependence of x_{ss} on M. The second part pertains to the "transient" displacement $\delta x_M \equiv x_M - x_{ss}$. Saddle-point integration, done by closely following the procedure detailed in Ref. [4], gives a closed-form solution for δx_M, the form of which depends on the region of parameter space under consideration. For parameters relevant to a linear collider, the bounding envelope of δx_M takes the form:

$$\frac{|\delta x_M|}{x_0} \simeq \left[\frac{\gamma(0)}{\gamma(\sigma)}\right]^{1/4}\frac{\sqrt{E}\ \exp\left[c(\eta)E - M\frac{\omega\tau}{2Q}\right]}{4M\sqrt{2\pi}\ |\sin(\omega\tau/2)|} \times \left\{\begin{array}{ll} 1/|1-\eta^2|^{1/4}\ ; & \eta \text{ not near } 1 \\ \left(\frac{4}{3}\right)^{1/6}\frac{\Gamma(1/3)}{\sqrt{2\pi}}\ E^{1/6}\ ; & \eta = 1. \end{array}\right. \tag{7}$$

The auxiliary relations comprising Eq. (7) are:

$$E(\sigma, M) = \Sigma(1)\left[\frac{w_0 q e \mathcal{L}^2}{\bar{\kappa}\gamma(0)mc^2}\,\frac{\Sigma(\sigma)}{\Sigma(1)}\,M\right]^{1/2} ;$$

$$\eta(\sigma, M) = \frac{\bar{\kappa}}{2E(\sigma, M)}\,|f_\gamma|\,\frac{\Sigma(\sigma)}{\Sigma(1)}\,\frac{M}{\mathcal{M}} ;$$

$$c(\eta) = \begin{cases} \frac{1}{2}\left[\sqrt{1-\eta^2} \;+\; \frac{1}{2\eta}\arctan\left(\frac{2\eta\sqrt{1-\eta^2}}{1-2\eta^2}\right)\right]; & \eta < 1 \\ \frac{\pi}{4\eta}; \quad \eta \geq 1; & \end{cases}$$

$$\Sigma(\sigma) = \int_0^\sigma d\sigma' \left[\frac{\gamma(0)}{\gamma(\sigma')}\right]^{1/2} \;=\; \frac{2\sqrt{\gamma(0)}\left[\sqrt{\gamma(\sigma)} - \sqrt{\gamma(0)}\right]}{\gamma(1)-\gamma(0)} ;$$

in which $\bar{\kappa}$ is the focusing strength averaged over the linac; $\mathcal{M}$ is the total number of bunches in the train; $|f_\gamma|$ is the magnitude of the total fractional energy spread across the bunch, or twice the total fractional focusing variation, and is constant along the linac; and the second equality for the generalized spatial coordinate Σ pertains to constant acceleration for which $\gamma \propto \sigma$, the case we use for numerical examples.

Fig. 2 illustrates good agreement between the envelope calculated analytically from Eq. (7) and bunch displacements calculated numerically from Eq. (1). We have also numerically solved a discrete version of Eq. (1) in which the cavities and focusing elements are localized entities; the solution overlaps that of Fig. 2. In addition, we considered separately the cases of linear variation of the focusing strength and of the beam energy; again, the numerical solutions closely agree for linear-collider parameters. In turn, Fig. 2 is a good indicator of the utility of the analytic solution for quantifying multibunch BBU in a linear collider.

The expression for $|\delta x_M|$ in Eq. (7) reflects a number of physical processes. The coefficient involving beam energy manifests adiabatic damping. The factor $|\sin(\omega\tau/2)|$ is a relic of a resonance function deriving from the coupling between the bunch spacing and the deflecting-mode frequency. Resonances lie near even-order wake zero-crossings [4]; because the solution is valid only away from zero-crossing, resonance is removed. The focusing variation represented by $|f_\gamma|$ regulates exponential growth, and finite Q yields exponential damping. An unphysical artifact of the saddle-point integration is also present, namely a localized singularity at $\eta=1$. In actuality $|\delta x_M|$ varies smoothly through the value provided in Eq. (7) for $\eta=1$; we analytically continue our plots by hand through this value. Yet "$\eta=1$" does have special physical significance: it demarks the onset of saturation of exponential growth and, with infinite Q, algebraic decay of the envelope. For $\eta \geq 1$ the "growth factor" $c(\eta)E$ is independent of bunch number M and of linac coordinate σ; temporal "damping" then ensues through a negative power of M, and spatial "damping" ensues adiabatically as already mentioned. Therefore $\eta=1$ demarks a global maximum in the envelope $|\delta x_M|$. The effect of the focusing variation is saturation of the exponential growth, not damping; its action distinctly differs from that of a real effective Q.

The special significance of $\eta=1$ translates into a criterion for the focusing variation to be effective. Specifically, one should choose a value of f_γ that ensures $\eta(1,\mathcal{M})>1$, *i.e.*, that $\eta=1$ is reached somewhere along the bunch train before it leaves the linac. According to the auxiliary relations to Eq. (7), the criterion is $|f_\gamma| > 2E(1,\mathcal{M})/\bar{\kappa}$.

The steady-state and transient displacements, being uncorrelated, comprise a measure of the total projected normalized emittance as $\varepsilon \equiv (|x_{ss}|^2 + |\delta x_M|^2_{max})\,\gamma\kappa/\mathcal{L}$, wherein $|x_{ss}| = x_0[\gamma(0)/\gamma(\sigma)]^{1/4}$ per Eq. (6), and $|\delta x_M|_{max}$ is the maximum value of the transient envelope reached along the bunch train. If $\eta<1$ always, then the maximum is reached at the last bunch $M=\mathcal{M}$. Otherwise, the maximum corresponds to the value of $|\delta x_M|$ at which $\eta=1$. Imposing a focusing variation will reduce the transient envelope, but it also will establish a harmonic variation of x_{ss} with M and thereby introduce a nonzero steady-state emittance ε_{ss}. For this reason the quantity of interest is the ratio $(\varepsilon-\varepsilon_{ss})/\varepsilon_{ss} = (|\delta x_M|_{max}/|x_{ss}|)^2$. This quantity, calculated from the analytic expressions given in Eqs. (6) and (7), is plotted against $|f_\gamma|$ in Fig. 3 for various values of w_0. Fig. 3 points to the region of parameter space that, respecting multibunch BBU, admits viable linear-collider designs. In particular it shows that to achieve low multibunch emittance without aid from a focusing variation requires small wake amplitudes. Otherwise, as depicted, a modest energy spread relieves the constraint on wake amplitude.

In summary, designing a linear collider involves trading between wake amplitude and energy spread (or focusing variation). For the NLC, the wake amplitude is ultimately determined by cell-to-cell coupling in the RDDS, which in turn relates to achievable fabrication tolerances, and to the efficacy of its higher-order-mode outcoupler. There are, of course, practical limitations on the energy spread, to include longitudinal beam requirements at the interaction point, lattice chromaticity, etc. Nonetheless, introducing a modest energy spread constitutes a backup in case sufficiently low wake amplitudes prove generally unfeasible.

The authors are grateful for stimulating discussions with M. Syphers, especially as concerns the interpretation of multibunch emittance, and with G. Stupakov, who provided specifics of the NLC design. This work was supported by the Universities Research Association, Inc., under contract DE-AC02-76CH00300 with the U.S. Department of Energy.

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TABLES

Table 1. Nominal Top-Level Linear-Collider Design Parameters

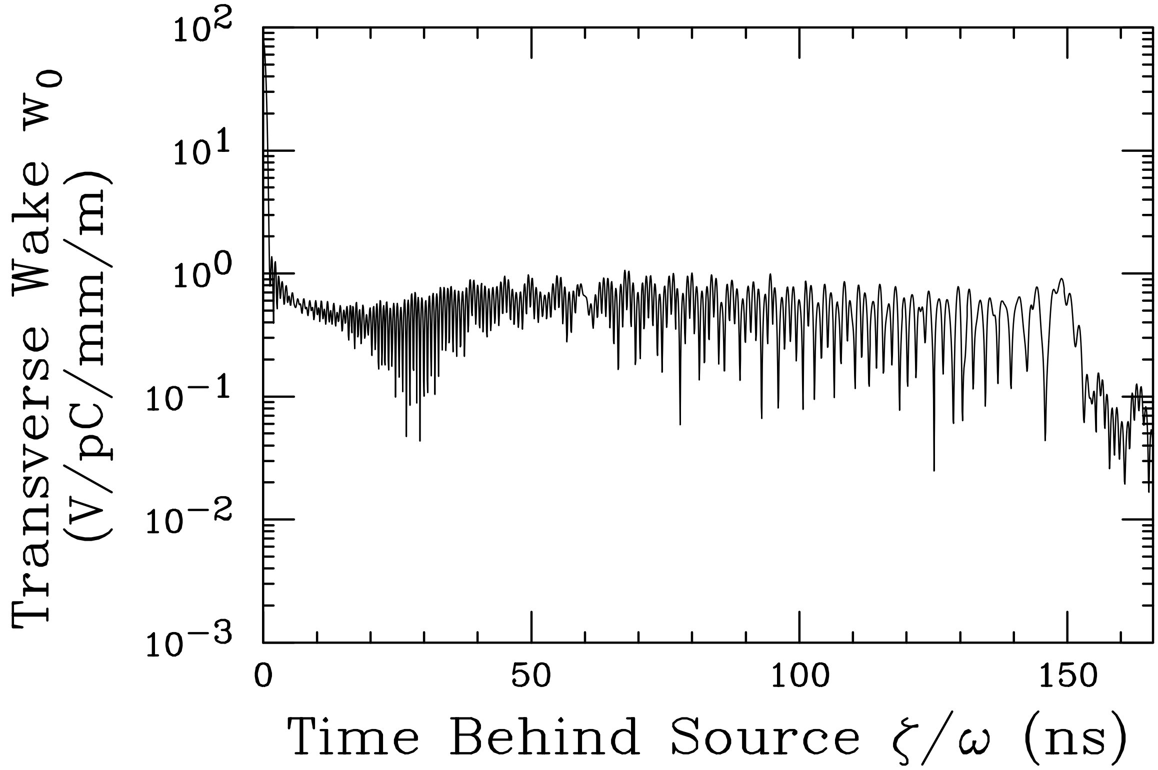
Parameter	Value
Total initial energy $\mathcal{E}_i = \gamma(0)mc^2$	10 GeV
Total final energy $\mathcal{E}_f = \gamma(1)mc^2$	1 TeV
Linac length $\mathcal{L}$	10 km
$\bar{\kappa} = 2\pi\times$ total no. betatron periods	$2\pi \times 100$
Bunch charge q	1 nC
No. bunches in train $\mathcal{M}$	90
Bunch spacing τ	2.8 ns
Deflecting-mode angular frequency ω	$2\pi \times 14.95$ GHz
Deflecting-mode quality factor Q	∞
Wake amplitude w_0	1 V/pC/mm/m

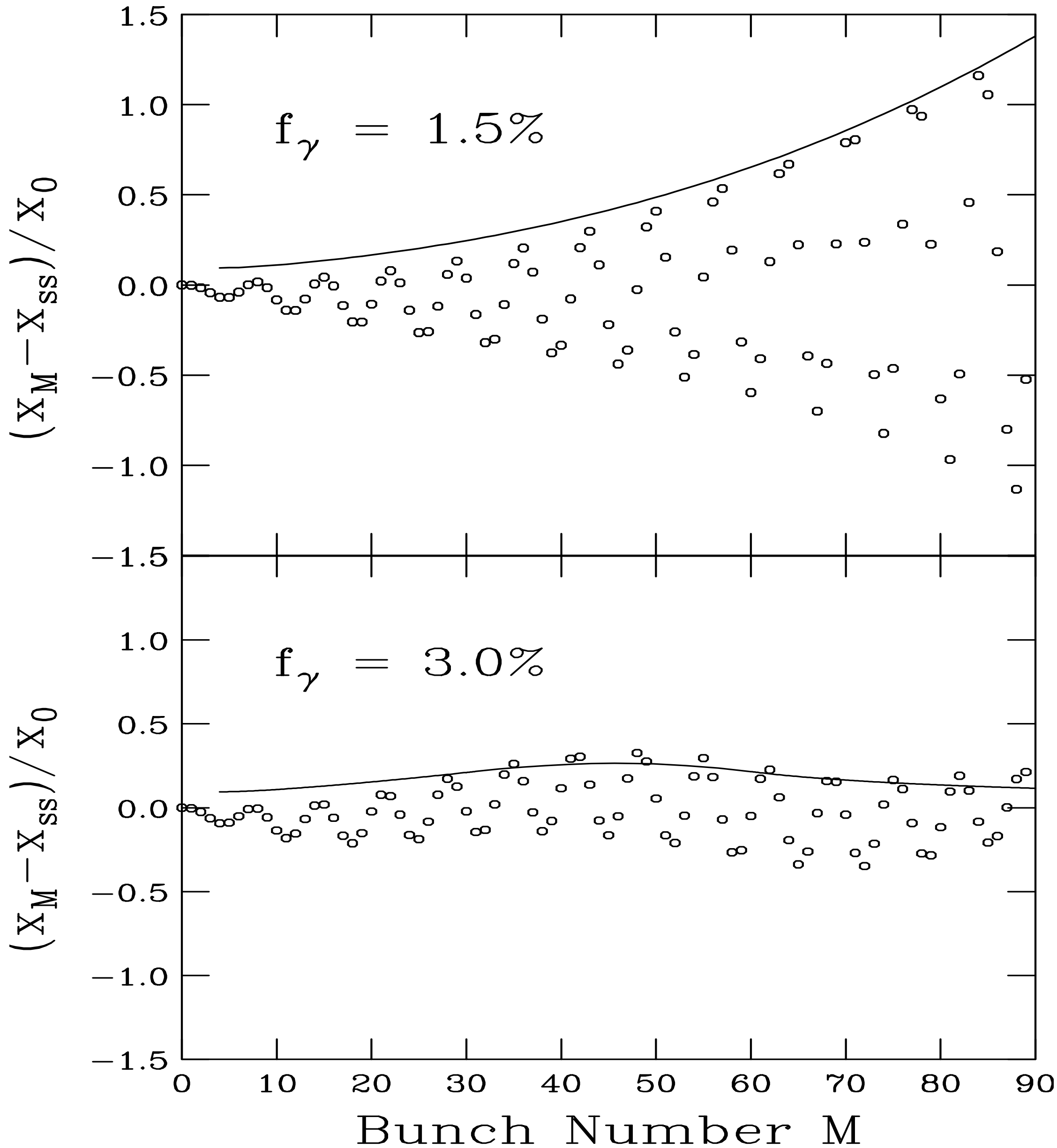
FIGURES

FIG. 1. Amplitude of deflecting wake in a prototypical RDDS. This plot is representative; the amplitude is sensitively dependent on construction tolerances and higher-order-mode outcoupling. It can be lower or higher than indicated here.

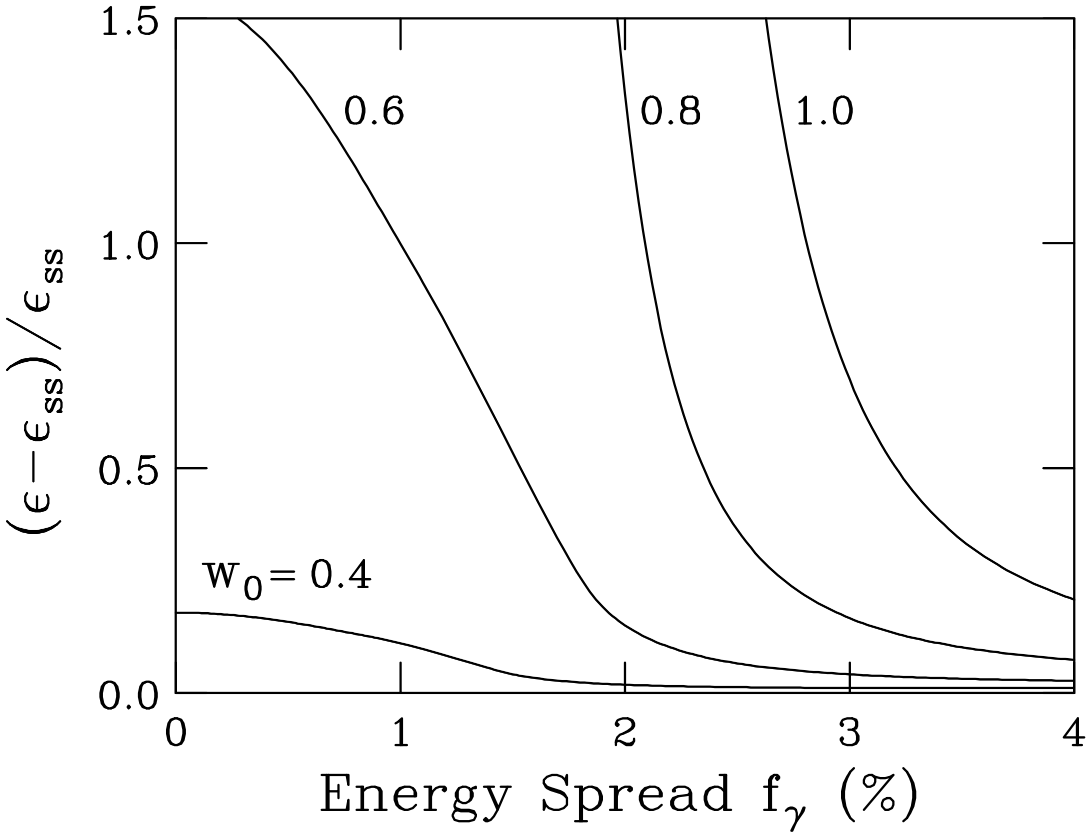
FIG. 2. Analytic envelope at the linac exit (solid curve) plotted against the transverse displacement of bunches calculated numerically. Inputs are per Table 1 with total energy spreads of 1.5% (top) and 3% (bottom).

FIG. 3. Total normalized transverse multibunch emittance at the linac exit, referenced to its steady-state value, vs. total energy spread across the bunch train, plotted for various wake amplitudes. Inputs are per Table 1.


Transverse Wake w_0
(V/pC/mm/m)
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
0
50
100
150
Time Behind Source ζ/ω (ns)



1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
f_γ = 1.5%
$(X_M - X_{SS})/X_0$
1.0
0.5
0.0
−0.5
−1.0
−1.5
f_γ = 3.0%
$(X_M - X_{SS})/X_0$
0
10
20
30
40
50
60
70
80
90
Bunch Number M



1.5
1.0
0.5
0.0
0
1
2
3
4
0.6
0.8
1.0
$w_0 = 0.4$
$(\epsilon - \epsilon_{ss})/\epsilon_{ss}$
Energy Spread f_γ (%)